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On the expansion of a function in a
Fourier series with prescribed phases

by

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§ 1. Introduction

In this report ¹⁾ we shall consider the following expansion of a given real function

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx - \frac{1}{2} \alpha_n \pi) \quad 1.1$$

in the halfperiod interval $0 \leq x \leq \pi$. The α_n are given constants, which will satisfy the conditions

$$\alpha_n = \alpha + O(n^{-\theta}), \quad \theta > 1$$

for $n \rightarrow \infty$ and

$$-1 < \operatorname{Re} \alpha < 1.$$

We shall also consider the expansion

$$f(x) = \sum_{n=0}^{\infty} a_n \sin(nx - \frac{1}{2} \alpha_n \pi) \quad 1.2$$

which differs from 1.1 in having a constant term.

These expansions have been investigated by H.A. LAUWERIER ²⁾ and G.W. VELTKAMP. The former obtained the following results. Expansion 1.1 is generally possible if it is required that $b_n \rightarrow \infty$ for $n \rightarrow \infty$. The coefficients b_n appear to be of subharmonic order $-1 + |\operatorname{Re} \alpha|$ for $n \rightarrow \infty$, expansion 1.2 is generally possible if it is required that $\sum_{n=0}^{\infty} |a_n| < \infty$. The coefficients a_n appear to be of hyperharmonic order $-1 - |\operatorname{Re} \alpha|$ for $n \rightarrow \infty$.

In the simple special case $\alpha_n = \alpha$ it is possible to determine the expansion coefficients in the following way.

To the set

$$\sin(nx - \frac{1}{2} \alpha \pi), \quad n \geq 1 \quad 3)$$

the following set of biorthogonal functions is associated

$$k_m(x) = 2 \tan^{\alpha} \frac{1}{2} x \sum_{k=1}^m e_{m-k} \sin kx, \quad m \geq 1 \quad 1.3$$

where the e_k are defined by the generating function

1) Research carried out under the direction of Prof. Dr. D. van Dantzig.

2) H.A. Lauwerier, On certain trigonometrical expansions. Report TW43 (1957) Mathematisch Centrum.

3) In Lauwerier's notation is $\alpha = -2\mu$.

$$\left(\frac{1+s}{1-s}\right)^\alpha = \sum_0^\infty e_k s^k \quad . \quad 1.4$$

The first few coefficients are

$$e_0 = 1 \quad e_1 = 2\alpha \quad e_2 = 2\alpha^2 \quad .$$

We also note the relation

$$\frac{2}{\pi} \int_0^\pi \tan^{-\alpha} \frac{1}{2}x \sin kx \, dx = -\sec \frac{1}{2}\alpha\pi e_k, \quad k \geq 1. \quad 1.5$$

In view of the orthogonality relation

$$\frac{1}{\pi} \int_0^\pi k_m(x) \sin(nx - \frac{1}{2}\alpha\pi) dx = \delta_{m,n}, \quad m, n \geq 1 \quad 1.6$$

the coefficients b_n can be written as

$$b_n = \frac{1}{\pi} \int_0^\pi f(x) k_n(x) dx, \quad n \geq 1. \quad 1.7$$

The more general expansion 1.1 can be reduced to a set of linear equations in the following way. Since the set

$$\sin(nx - \frac{1}{2}\alpha\pi), \quad n \geq 1,$$

is asymptotically orthogonal to the set $k_m(x)$, $m \geq 1$, it seems appropriate to determine the coefficients b_n from

$$\sum_{n=1}^\infty C_{m,n} b_n = \frac{1}{\pi} \int_0^\pi f(x) k_m(x) dx, \quad m \geq 1, \quad 1.8$$

where

$$C_{m,n} = \frac{1}{\pi} \int_0^\pi k_m(x) \sin(nx - \frac{1}{2}\alpha\pi) dx, \quad m, n \geq 1. \quad 1.9$$

It will be shown in §4 that the $C_{m,n}$ can be expressed in terms of coefficients

$$e_{m,n} = \frac{1}{\pi} \int_0^\pi k_m(x) \sin(nx + \frac{1}{2}\alpha\pi) dx, \quad m \geq 1, \quad n \text{ integer}. \quad 1.10$$

Comparison with 1.5 shows that

$$e_{m,n} = -\delta_{m,-n} \text{ for } n \leq -1. \quad 1.11$$

For these coefficients the following relations will be proved

$$(-1)^n e_{m,n} = (-1)^m e_{n,m}, \quad 1.12$$

$$m(e_{m+1,n} - e_{m-1,n}) + (n+1)e_{m,n+1} - (n-1)e_{m,n-1} = 0, \quad 1.13$$

$$e_{m+1,n} + e_{m-1,n} - e_{m,n+1} - e_{m,n-1} = e_{m,0} e_{1,n}, \quad 1.14$$

$$e_{m,0} = e_m, \quad 1.15$$

$$e_{m,n} = m \sum_{k=0}^n (-1)^k \frac{(m+k-1)!}{k!(n-k)!(m-n+k)!} e_{m-n+2k}. \quad 1.16$$

Since the coefficients $e_{m,n}$ and e_n depend on α , we shall sometimes write $e_{m,n}(\alpha)$ and $e_n(\alpha)$. For coefficients with opposite α we shall derive

$$e_{m,n}(-\alpha) = (-1)^{m+n} e_{m,n}(\alpha). \quad 1.17$$

The rapidly convergent expansion 1.2 may be treated in a similar way. We first consider the case $\alpha_n = \alpha$. Then to the set

$$\sin(nx - \frac{1}{2}\alpha\pi), \quad n \geq 0,$$

the following set of biorthogonal functions is associated

$$h_m(x) = -\tan^{\alpha-1} \frac{1}{2}x \sum_{k=0}^m \varepsilon_k e_{m-k}(\alpha-1) \cos kx, \quad m \geq 0 \quad 1.18$$

if $0 < \alpha < 1$ and

$$h_m(x) = \tan^{\alpha+1} \frac{1}{2}x \sum_{k=0}^m k e_{m-k}(\alpha+1) \cos kx, \quad m \geq 0 \quad 1.19$$

if $-1 < \alpha < 0$. ($\varepsilon_0=1$, $\varepsilon_k=2$ ($k \geq 1$)).

In view of the orthogonality relation

$$\frac{1}{\pi} \int_0^\pi h_m(x) \sin(nx - \frac{1}{2}\alpha\pi) dx = \delta_{m,n}, \quad m, n \geq 0, \quad 1.20$$

the expansion coefficients a_n can be written as

$$a_n = \frac{1}{\pi} \int_0^\pi f(x) h_n(x) dx, \quad n \geq 0. \quad 1.21$$

In the case of the more general expansion 1.2 the expansion coefficients again can be determined from a set of linear equations viz.

$$\sum_{n=0}^{\infty} d_{m,n} a_n = \frac{1}{\pi} \int_0^\pi f(x) h_m(x) dx, \quad m \geq 1, \quad 1.22$$

where

$$d_{m,n} = \frac{1}{\pi} \int_0^{\pi} h_m(x) \sin(nx - \frac{1}{2} \alpha_n \pi) dx, \quad m \geq 0. \quad 1.23$$

As in the previous case the $d_{m,n}$ can be expressed in terms of coefficients

$$f_{m,n} = \frac{1}{\pi} \int_0^{\pi} h_m(x) \sin(nx + \frac{1}{2} \alpha \pi) dx, \quad m \geq 0, \quad 1.24$$

but these coefficients will be shown to be related to the $e_{m,n}$ according to

$$\begin{aligned} f_{m,n}(\alpha) &= -e_{n,m}(1-\alpha), \quad 1 > \alpha > 0, \\ f_{m,n}(\alpha) &= -e_{n,m}(-1-\alpha), \quad -1 < \alpha < 0. \end{aligned} \quad 1.25$$

§ 2. The coefficients $e_{m,n}$.

In this section we prove formulae 1.12 - 1.17. We first derive a generating function for the $e_{m,n}$. According to 1.6 and 1.10 we have

$$\begin{aligned} \frac{1}{\pi} \int_0^{\pi} k_m(x) \sin(nx - \frac{1}{2} \alpha \pi) dx &= \delta_{m,n}, \quad m, n \geq 1, \\ \frac{1}{\pi} \int_0^{\pi} k_m(x) \sin(nx + \frac{1}{2} \alpha \pi) dx &= e_{m,n}, \quad m \geq 1. \end{aligned}$$

Addition and subtraction yields

$$\begin{aligned} \frac{2}{\pi} \int_0^{\pi} k_m(x) \sin nx \, dx &= \sec \frac{1}{2} \alpha \pi (e_{m,n} + \delta_{m,n}), \quad m, n \geq 1, \\ \frac{\xi_n}{\pi} \int_0^{\pi} k_m(x) \cos nx \, dx &= \csc \frac{1}{2} \alpha \pi (e_{m,n} - \delta_{m,n}), \quad m \geq 1, n \geq 0. \end{aligned}$$

Hence we have

$$\begin{aligned} k_m(x) &= \sec \frac{1}{2} \alpha \pi \sum_{n=1}^{\infty} (e_{m,n} + \delta_{m,n}) \sin nx, \quad m \geq 1, \\ &= \csc \frac{1}{2} \alpha \pi \sum_{n=0}^{\infty} (e_{m,n} - \delta_{m,n}) \cos nx, \quad m \geq 1. \end{aligned}$$

From this formula we find the generating function ⁴⁾

$$\cos(mx - \frac{1}{2} \alpha \pi) = \sum_{n=0}^{\infty} e_{m,n} \cos(nx + \frac{1}{2} \alpha \pi). \quad 2.1$$

4) This generating function is due to a suggestion of H.A. Lauwerier.

This formula has been obtained for $m \geq 1$. By means of this expression the $e_{m,n}$ can also be defined for $m \leq 0$, $n \geq 0$. Then we have

$$e_{m,n} = \delta_{-m,n}, \quad m \leq n, \quad n \geq 0. \quad 2.2$$

If, in 2.1 we replace x by $\pi - x$, we find

$$\cos(mx + \frac{1}{2}\alpha\pi) = \sum_{n=0}^{\infty} (-1)^{m+n} e_{m,n} \cos(nx - \frac{1}{2}\alpha\pi). \quad 2.3$$

Comparison of 2.1 with 2.3 immediately yields 1.17. Relation 1.14 can be proved as follows

$$\begin{aligned} & \sum_{n=0}^{\infty} (e_{m+1,n} + e_{m-1,n} - e_{m,n+1} - e_{m,n-1}) \cos(nx + \frac{1}{2}\alpha\pi) \\ &= \sum_{n=0}^{\infty} (e_{m+1,n} + e_{m-1,n}) \cos(nx + \frac{1}{2}\alpha\pi) \\ & \quad - \sum_{n=0}^{\infty} e_{m,n} \cos\{(n-1)x + \frac{1}{2}\alpha\pi\} - \sum_{n=0}^{\infty} e_{m,n} \cos\{(n+1)x + \frac{1}{2}\alpha\pi\} \\ &= \cos\{(m+1)x - \frac{1}{2}\alpha\pi\} + \cos\{(m-1)x - \frac{1}{2}\alpha\pi\} \\ & \quad - 2 \cos x \sum_{n=0}^{\infty} e_{m,n} \cos(nx + \frac{1}{2}\alpha\pi) + e_{m,0} \cos(x - \frac{1}{2}\alpha\pi) \\ &= e_{m,0} \sum_{n=0}^{\infty} e_{1,n} \cos(nx + \frac{1}{2}\alpha\pi) \end{aligned}$$

because of $e_{m,-1} = 0$ (comp. 1.10).

Identification of corresponding coefficients yields 1.14.

Differentiation of 2.1 and 2.3 yields

$$m \sin(mx - \frac{1}{2}\alpha\pi) = \sum_{n=1}^{\infty} n e_{m,n} \sin(nx + \frac{1}{2}\alpha\pi) \quad 2.4$$

$$m \sin(mx + \frac{1}{2}\alpha\pi) = \sum_{n=1}^{\infty} n (-1)^{m+n} e_{m,n} \sin(nx - \frac{1}{2}\alpha\pi). \quad 2.5$$

Multiply 2.5 with $k_p(x)/\pi$ and integrate from 0 to π . We find

$$m e_{p,m} = n (-1)^{m+n} e_{m,n} \delta_{p,n} = p (-1)^{m+p} e_{m,p}$$

in accordance with 1.12.

Relation 1.13 can be proved by means of 2.1 and 2.4. We have

$$\begin{aligned} & \sum_{n=0}^{\infty} m (e_{m+1,n} - e_{m-1,n}) \cos(nx + \frac{1}{2}\alpha\pi) \\ &= m \cos\{(m+1)x - \frac{1}{2}\alpha\pi\} - m \cos\{(m-1)x - \frac{1}{2}\alpha\pi\} \end{aligned}$$

$$\begin{aligned}
 &= -2m \sin x \sin(mx - \frac{1}{2}\alpha\pi) \\
 &= -2 \sin x \sum_{n=1}^{\infty} n e_{m,n} \sin(nx + \frac{1}{2}\alpha\pi) \\
 &= \sum_{n=1}^{\infty} n e_{m,n} \cos\{(n+1)x + \frac{1}{2}\alpha\pi\} - \sum_{n=1}^{\infty} n e_{m,n} \cos\{(n-1)x + \frac{1}{2}\alpha\pi\} \\
 &= \sum_{n=0}^{\infty} \left\{ (n-1)e_{m,n-1} - (n+1)e_{m,n+1} \right\} \cos(nx + \frac{1}{2}\alpha\pi),
 \end{aligned}$$

because of $e_{m,-1}=0$.

Identification of corresponding coefficients yields 1.13.

For the proof of 1.15 we first show

$$\sum_{k=0}^m e_{m-k}(\alpha) e_k(-\alpha) = \delta_{m,0} \quad . \quad 2.6$$

Indeed we have, because of 1.4,

$$\begin{aligned}
 1 &= \left(\frac{1+s}{1-s}\right)^{\alpha} \left(\frac{1+s}{1-s}\right)^{-\alpha} \\
 &= \sum_{m=0}^{\infty} e_m(\alpha) s^m \sum_{k=0}^{\infty} e_k(-\alpha) s^k \\
 &= \sum_{m=0}^{\infty} s^m \sum_{k=0}^m e_{m-k}(\alpha) e_k(-\alpha).
 \end{aligned}$$

Using 1.10, 1.3, 1.5 and 2.6 we have

$$\begin{aligned}
 e_{m,0} &= \sin \frac{1}{2}\alpha\pi \frac{1}{\pi} \int_0^{\pi} k_m(x) dx \\
 &= \sin \frac{1}{2}\alpha\pi \sum_{k=1}^m e_{m-k}(\alpha) \frac{2}{\pi} \int_0^{\pi} \tan^{\alpha} \frac{1}{2}x \sin kx dx \\
 &= - \sum_{k=1}^m e_{m-k}(\alpha) e_k(-\alpha) \\
 &= e_m,
 \end{aligned}$$

because $e_0=1$.

We shall prove 1.16 by induction. By substitution it is verified, that 1.16 satisfies the recurrence relation 1.13. Moreover for $n=0$ 1.16 becomes identical with 1.15 which we already know to be true. For $n=-1$ we find from 1.16 $e_{m,-1}=0$ in accordance with 1.11. Hence, since 1.16 is valid for two values of n , it must be valid for all values of n .

We conclude this section with a remark concerning the practical evaluation of the $e_{m,n}$. Let it be required to evaluate a square array of $e_{m,n}$, $m,n=1\dots N$. We start with the column $e_{m,0}=e_m$. For the e_m Lauwerier proved the recurrence relation

$$me_m = 2\alpha e_{m-1} + (m-2)e_{m-2} \quad 2.7$$

which admits the evaluation of all e_m from $e_0=1$ and $e_1=2\alpha$. Then the $e_{m,1}$ can be evaluated by aid of 1.13, except $e_{N,1}$. However, from 1.13 and 1.14 we obtain

$$(m+n+1)e_{m,n+1} = 2m e_{m-1,n} - (m-n+1)e_{m,n-1} - m e_{m,0}e_m \quad 2.8$$

and this formula can be used for evaluating $e_{N,1}$. In general we evaluate $e_{m,n}$, $m \geq n$, $m < N$ from two preceding columns by aid of 1.13. For $m < n$ we use 1.12 and $e_{N,n}$ is found by aid of 2.8.

A table of $e_{m,n}$, $m,n=0(1)6$ is added to this report.

§ 3. The coefficients $f_{m,n}$

According to 1.20 and 1.22 we have

$$\begin{aligned} \frac{1}{\pi} \int_0^\pi h_m(x) \sin(nx - \frac{1}{2}\alpha\pi) dx &= \delta_{m,n}, \quad m,n \geq 0, \\ \frac{1}{\pi} \int_0^\pi h_m(x) \sin(nx + \frac{1}{2}\alpha\pi) dx &= f_{m,n}, \quad m \geq 0. \end{aligned}$$

In a similar way as before we find

$$\cos(mx - \frac{1}{2}\alpha\pi) = \sum_{n=0}^{\infty} f_{m,n} \cos(nx + \frac{1}{2}\alpha\pi), \quad m \geq 0 \quad 3.1$$

Furthermore we have, because of 2.5 and 1.12,

$$\begin{aligned} \sin(mx + \frac{1}{2}\alpha\pi) &= \sum_{n=1}^{\infty} \frac{n}{m} (-1)^{m+n} e_{m,n}(\alpha) \sin(nx - \frac{1}{2}\alpha\pi) \\ &= \sum_{n=1}^{\infty} e_{n,m}(\alpha) \sin(nx - \frac{1}{2}\alpha\pi) \end{aligned} \quad 3.2$$

If $0 < \alpha < 1$ we put $\alpha = 1 - \alpha'$. Substitution in 3.2 gives, if the primes are dropped,

$$\cos(mx - \frac{1}{2}\alpha\pi) = - \sum_{n=1}^{\infty} e_{n,m}(1-\alpha) \cos(nx + \frac{1}{2}\alpha\pi). \quad 3.3$$

Comparison of 3.1 and 3.3 yields

$$f_{m,n}(\alpha) = -e_{n,m}(1-\alpha) , \quad 1 > \alpha > 0.$$

If $-1 < \alpha < 0$ we put $\alpha = -1 - \alpha'$ and find in the same way

$$f_{m,n}(\alpha) = -e_{n,m}(-1-\alpha) \quad 0 > \alpha > -1 .$$

Hence the relations 1.25 are proved

§ 4. The general expansions

We shall show how the expansion coefficients b_n in

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx - \frac{1}{2} \alpha_n \pi)$$

can be determined.

Let $\alpha_n \rightarrow \alpha$ for $n \rightarrow \infty$. By means of the trigonometrical identity

$$\sin(\varphi - \beta) = \frac{\sin(\beta + \gamma)}{\sin 2\gamma} \left[\sin(\varphi - \gamma) + \frac{\sin(\beta - \gamma)}{\sin(\beta + \gamma)} \sin(\varphi + \gamma) \right] ,$$

this can be written in the form

$$f(x) = \sum_{n=1}^{\infty} b'_n \left[\sin(nx - \frac{1}{2} \alpha \pi) + s_n \sin(nx + \frac{1}{2} \alpha \pi) \right] , \quad 4.1$$

where

$$b'_n = \frac{\sin \frac{1}{2}(\alpha + \alpha_n)}{\sin \alpha \pi} b_n , \quad 4.2$$

$$s_n = \frac{\sin \frac{1}{2}(\alpha - \alpha_n)}{\sin \frac{1}{2}(\alpha + \alpha_n)} . \quad 4.3$$

Proceeding as indicated in the introduction we find

$$\sum_{n=1}^{\infty} \left\{ \sigma_{m,n} + s_n e_{m,n}(\alpha) \right\} b'_n = \frac{1}{\pi} \int_0^{\pi} f(x) k_m(x) dx. \quad 4.4$$

In the same way we can derive for the rapidly convergent expansion 1.2 the sets of linear equations

$$\sum_{n=0}^{\infty} \left\{ \sigma_{m,n} - s_n e_{n,m}(1-\alpha) \right\} a'_n = \frac{1}{\pi} \int_0^{\pi} f(x) h_m(x) dx, \quad 0 < \alpha < 1, \quad 4.5$$

$$\sum_{n=0}^{\infty} \left\{ \sigma_{m,n} - s_n e_{n,m}(-1-\alpha) \right\} a'_n = \frac{1}{\pi} \int_0^{\pi} f(x) h_m(x) dx, \quad -1 < \alpha < 0, \quad 4.6$$

where

$$a'_n = \frac{\sin \frac{1}{2}(\alpha + \alpha_n)}{\sin \alpha \pi} a_n .$$

If for $f(x)$ a Fourier expansion valid in the half-period interval $0 < x < \pi$ is known, we can evaluate the right hand members of 4.4, 4.5 or 4.6 in a similar way. Let e.g.

$$f(x) = \sum_{n=1}^{\infty} p_n \sin(nx - \frac{1}{2} \varphi_n \pi)$$

then

$$\frac{1}{\pi} \int_0^{\pi} f(x) k_m(x) dx = \sum_{n=1}^{\infty} p'_n \left\{ \delta_{m,n} + q_n e_{m,n}(\alpha) \right\}$$

where

$$p'_n = \frac{\sin \frac{1}{2}(\alpha + \varphi_n)}{\sin \alpha \pi} p_n ,$$

$$q_n = \frac{\sin \frac{1}{2}(\alpha - \varphi_n) \pi}{\sin \frac{1}{2}(\alpha + \varphi_n) \pi} .$$

Table of $e_{m,n}(\alpha)$

$$e_{00} = 1$$

$$e_{10} = 2\alpha$$

$$e_{20} = 2\alpha^2$$

$$e_{30} = \frac{2}{3}\alpha(1+2\alpha^2)$$

$$e_{40} = \frac{2}{3}\alpha(2+\alpha^2)$$

$$e_{50} = \frac{2}{15}\alpha(3+10\alpha^2+2\alpha^4)$$

$$e_{60} = \frac{2}{45}\alpha^2(23+20\alpha^2+2\alpha^4)$$

$$e_{01} = 0$$

$$e_{11} = 1-2\alpha^2$$

$$e_{21} = \frac{8}{3}\alpha(1-\alpha^2)$$

$$e_{31} = 2\alpha^2(1-\alpha^2)$$

$$e_{41} = \frac{16}{15}\alpha(1-\alpha^2)(1+\alpha^2)$$

$$e_{51} = \frac{2}{9}\alpha^2(1-\alpha^2)(7+2\alpha^2)$$

$$e_{61} = \frac{8}{105}\alpha(1-\alpha^2)(9+16\alpha^2+2\alpha^4)$$

$$e_{02} = 0$$

$$e_{12} = -\frac{4}{3}\alpha(1-\alpha^2)$$

$$e_{22} = 1-2\alpha^2(2-\alpha^2)$$

$$e_{32} = \frac{4}{5}\alpha(1-\alpha^2)(3-2\alpha^2)$$

$$e_{42} = \frac{8}{9}\alpha^2(1-\alpha^2)^2$$

$$e_{52} = \frac{4}{21}\alpha(1-\alpha^2)(5-2\alpha^2-2\alpha^4)$$

$$e_{62} = \frac{2}{15}\alpha^2(1-\alpha^2)^2(6+\alpha^2)$$

$$e_{03} = 0$$

$$e_{13} = \frac{2}{3}\alpha^2(1-\alpha^2)$$

$$e_{23} = -\frac{8}{15}\alpha(1-\alpha^2)(3-2\alpha^2)$$

$$e_{33} = 1-\frac{2}{9}\alpha^2(19-14\alpha^2+4\alpha^4)$$

$$e_{43} = \frac{16}{63}\alpha(1-\alpha^2)(9-5\alpha^2+2\alpha^4)$$

$$e_{53} = \frac{2}{9}\alpha^2(1-\alpha^2)(3-\alpha^2+\alpha^4)$$

$$e_{63} = \frac{8}{405}\alpha(1-\alpha^2)(45-14\alpha^2+10\alpha^4+4\alpha^6)$$

$$e_{04} = 0$$

$$e_{14} = -\frac{4}{15}\alpha(1-\alpha^2)(1+\alpha^2)$$

$$e_{24} = \frac{4}{9}\alpha^2(1-\alpha^2)^2$$

$$e_{34} = -\frac{4}{21}\alpha(1-\alpha^2)(9-5\alpha^2+2\alpha^4)$$

$$e_{44} = 1-\frac{2}{9}\alpha^2(20-14\alpha^2+4\alpha^4-\alpha^6)$$

$$e_{54} = \frac{4}{81}\alpha(1-\alpha^2)(45-29\alpha^2+4\alpha^4-2\alpha^6)$$

$$e_{64} = \frac{4}{225}\alpha^2(1-\alpha^2)^2(27+4\alpha^2+2\alpha^4)$$

$$e_{05} = 0$$

$$e_{15} = \frac{2}{45} \alpha^2 (1 - \alpha^2) (7 + 2\alpha^2)$$

$$e_{25} = -\frac{8}{105} \alpha (1 - \alpha^2) (5 - 2\alpha^2 - 2\alpha^4)$$

$$e_{35} = \frac{2}{15} \alpha^2 (1 - \alpha^2) (3 - \alpha^2 + \alpha^4)$$

$$e_{45} = -\frac{16}{405} \alpha (1 - \alpha^2) (45 - 29\alpha^2 + 4\alpha^4 - 2\alpha^6)$$

$$e_{55} = 1 - \frac{2}{225} \alpha^2 (509 - 390\alpha^2 + 112\alpha^4 - 10\alpha^6 + 4\alpha^8)$$

$$e_{65} = \frac{8}{2475} \alpha (1 - \alpha^2) (675 - 414\alpha^2 + 136\alpha^4 + 4\alpha^6 + 4\alpha^8)$$

$$e_{06} = 0$$

$$e_{16} = -\frac{4}{315} \alpha (1 - \alpha^2) (9 + 16\alpha^2 + 2\alpha^4)$$

$$e_{26} = \frac{2}{45} \alpha^2 (1 - \alpha^2) (6 - 5\alpha^2 - \alpha^4)$$

$$e_{36} = -\frac{4}{405} \alpha (1 - \alpha^2) (45 - 14\alpha^2 + 10\alpha^4 + 4\alpha^6)$$

$$e_{46} = \frac{8}{675} \alpha^2 (1 - \alpha^2) (27 + 4\alpha^2 + 2\alpha^4)$$

$$e_{56} = -\frac{4}{1485} \alpha (1 - \alpha^2) (675 - 414\alpha^2 + 136\alpha^4 + 4\alpha^6 + 4\alpha^8)$$

$$e_{66} = 1 - \frac{2}{2025} \alpha^2 (4662 - 3499\alpha^2 + 1096\alpha^4 - 222\alpha^6 - 8\alpha^8 - 4\alpha^{10}).$$